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On best uniform restricted range approximation in complex-valued continuous function spaces[☆]

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Abstract

We investigate the problem of best restricted range approximation of complex-valued continuous functions for a very general system of restrictions. Our results, including the characterizations, uniqueness and strong uniqueness, extend all recent results due to Smirnovs. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

Let $C(Q)$ denote the Banach space of all complex-valued continuous functions on a compact Hausdorff space Q endowed with the uniform norm

$$\|f\| = \max_{t \in Q} |f(t)| \quad \forall f \in C(Q).$$

In the spirit of the best restricted range approximation in a real-valued continuous function space, see for example [2,5,14,15] and the relevant references therein, Smirnov and Smirnov [9,10] presented and formulated the problem of best restricted range approximations in a complex-valued continuous function space. The setting is as follows. Let P be a finite-dimensional subspace of $C(Q)$ and $\Omega = \{\Omega_t; t \in Q\}$ be a system of nonempty convex closed sets in the complex plane \mathbb{C} . Set

$$P_\Omega = \{p \in P : p(t) \in \Omega_t \text{ for all } t \in Q\}.$$

The problem considered here is to find an element $p^* \in P_\Omega$, which is called a best (restricted range) approximation to $f \in C(Q)$ from P_Ω , such that

$$\|f - p^*\| = \inf_{p \in P_\Omega} \|f - p\|.$$

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As pointed out in [10], this problem for a general class of restrictions is quite difficult. Therefore, in [9,10], Ω was assumed to be the system of the closed disks with the center $u(t)$ and radius $r(t) > 0$ for each $t \in Q$, that is,

$$\Omega_t = \{z \in \mathbb{C} : |z - u(t)| \leq r(t)\} \quad \forall t \in Q,$$

where $u, r \in C(Q)$. Under this assumption, the authors gave the results on existence, characterization, uniqueness and strong uniqueness of the best restricted range approximation. Recently, these results have been extended by Smirnovs [11–13] to a general restriction system Ω for which every Ω_t is a closed, strictly convex set with nonempty interior and “smooth” boundary, and, in addition, Ω_t is continuous relative to $t \in Q$ under the Hausdorff metric of sets.

In the present paper, we consider the same problem for a more general class of restrictions. More precisely, for any $t \in Q$, Ω_t is only assumed to have nonempty interior. Of course, it is natural to require that Ω_t have some continuity relative to $t \in Q$. Note that any closed convex subset can be expressed as a level set of a convex function. In fact, for any $t \in Q$, there exists a real convex function $F(\cdot, t)$ on \mathbb{C} such that

$$\partial\Omega_t = \{z \in \mathbb{C} : F(z, t) = 0\} \quad \forall t \in Q, \quad (1.1)$$

$$\text{int } \Omega_t = \{z \in \mathbb{C} : F(z, t) < 0\} \quad \forall t \in Q, \quad (1.2)$$

where $\partial\Omega_t$ and $\text{int } \Omega_t$ denote the boundary and interior of Ω_t , respectively. Thus, we assume that the required continuity for Ω to satisfy is that the function $F(\cdot, \cdot)$ continues on the product space $\mathbb{C} \times Q$. In this case, we establish some results on characterization, uniqueness and strong uniqueness, similar to but more general than the ones due to Smirnovs [9–13].

It should be remarked that our system of restrictions in the present paper, compared with Smirnovs’ systems in [11–13], is much more general since, for each $t \in Q$, the closed convex set Ω_t only needs to have nonempty interior, which, in fact, can be unbounded. In addition, for a system Ω satisfying Hypothesis 2.1, if the set-valued mapping $t \mapsto \Omega_t$ is continuous under the Hausdorff metric, we can verify that there exist continuous functions $F(\cdot, \cdot)$ on $\mathbb{C} \times Q$, which is convex with respect to the first variable, such that (1.1) and (1.2) hold. But the converse is obviously not true.

2. Preliminaries

In the paper, we assume that Q contains at least $n + 1$ points and that $\{\phi_1, \phi_2, \dots, \phi_n\} \subset P$ is a base of P , that is, any element $p \in P$ has a unique expression of the form

$$p = \sum_{i=1}^n c_i \phi_i,$$

where $c_i \in \mathbb{C}$, $i = 1, 2, \dots, n$.

As in [10–12], we need a basic hypothesis and some notations.

Hypothesis 2.1. Assume that there exists $p_0 \in P$ such that

$$p_0(t) \in \text{int } \Omega_t \quad \forall t \in Q. \quad (2.1)$$

For $f \in C(Q)$, $A \subseteq Q$, following [10], we define

$$P_{A,\Omega} = \{p \in P: p(t) \in \Omega_t, \forall t \in A\},$$

$$E_A(f, P_{B,\Omega}) = \inf\{\|f - p\|_A: p \in P_{B,\Omega}\},$$

where $\|f\|_A = \sup\{|f(t)|: t \in A\}$. In particular, we set, for short, $E_A(f, P_{B,\Omega}) = E_A(f)$ when $A = B$ and $E_A(f, P_{B,\Omega}) = E(f)$ when $A = B = Q$.

In general, we have $E_A(f) \leq E_B(f) \leq E(f)$ if $A \subseteq B \subseteq Q$.

Definition 2.1 (Smirnov and Smirnov [10]). An element $p^* \in P_{A,\Omega}$ is called a best (restricted range) approximation to f on A from $P_{A,\Omega}$ if

$$\|f - p^*\|_A = E_A(f).$$

In particular, a best (restricted range) approximation to f on Q from P_Ω is called a best (restricted range) approximation to f from P_Ω , for short.

Definition 2.2. A subset $A \subseteq Q$ is called an admissible subset for f with respect to P_Ω , if $E_A(f) = E(f)$.

Remark 2.1. If A is an admissible subset for f with respect to P_Ω , then any subset of Q containing A is also an admissible subset for f with respect to P_Ω .

Remark 2.2. Recall that (A, B) with $A \subseteq Q$, $B \subseteq Q$ is called an admissible pair for f with respect to P_Ω if $E_A(f, P_{B,\Omega}) = E(f)$, see, e.g., [10]. Clearly, if A is an admissible subset for f with respect to P_Ω , then the pair (A, A) is an admissible pair for f with respect to P_Ω . Conversely, if the pair (A, B) is an admissible pair for f with respect to P_Ω , then the set $A \cup B$ is an admissible subset for f with respect to P_Ω .

Remark 2.3. With almost the same arguments as the proof of Theorem 3.2 in [10], we can show that, for any $f \in C(Q)$, there exists at least one admissible subset A for f with respect to P_Ω such that the cardinality $|A| \leq 2n + 1$.

Finally, we need the concepts of the subdifferential and directional derivative of a real function.

Definition 2.3 (Rockafellar [6]). Let F be a convex function defined on \mathbb{C} and $z, u \in \mathbb{C}$. The subdifferential of F at z , denoted by $\partial F(z)$, is defined by

$$\partial F(z) = \{u \in \mathbb{C} : F(v) \geq F(z) + \operatorname{Re}(v - z)\bar{u} \ \forall v \in \mathbb{C}\},$$

while the directional derivative of F at z with respect to u , denoted by $F'(z)(u)$, is defined by

$$F'(z)(u) = \lim_{t \rightarrow +0} \frac{F(z + tu) - F(z)}{t}.$$

As is well-known [6], if F is convex then $\partial F(z)$ is a nonempty closed convex set in \mathbb{C} and

$$F'(z)(u) = \max \operatorname{Re} \overline{\partial F(z)}u.$$

The following proposition, which is a direct consequence of the definitions, is useful in the rest.

Proposition 2.1. *Let $z^* \in \mathbb{C}$ satisfy that $F(z^*) = 0$ and $z \in \mathbb{C}$. If $F(z) \leq 0 (< 0)$, then*

$$F'(z^*)(z - z^*) = \max \operatorname{Re} \overline{\partial F(z^*)}(z - z^*) \leq 0 (< 0). \tag{2.2}$$

3. Characterization of the best approximation

Let $f \in C(Q)$, $p^* \in P_\Omega$. Following [9,10], we define

$$M(f) = \{t \in Q : |f(t)| = \|f\|\}, \quad B(p^*) = \{t \in Q : p^*(t) \in \partial \Omega_t\}$$

and

$$\sigma_1(t) = f(t) - p^*(t) \quad \forall t \in Q,$$

From the continuity, it follows that $M(f)$ and $B(p^*)$ are compact. Furthermore, we define a set-valued mapping $\sigma_2(t)$ by

$$\sigma_2(t) = -\partial F(p^*(t), t) \quad \forall t \in Q,$$

where $\partial F(p^*(t), t)$ denotes the subdifferential of the function $F(\cdot, t)$ at $p^*(t)$.

The following proposition is the well-known Kolmogorov-type characterization of the best approximation from a convex subset of $C(Q)$, see, for example [1].

Proposition 3.1. *Let A be a nonempty closed subset of Q and G a closed non-empty convex subset of $C(A)$. Let $f \in C(A)$, $p^* \in G$. Then p^* is a best approximation to f from G if and only if*

$$\max_{t \in M_A(f - p^*)} \operatorname{Re}(p^* - p)(t) \overline{\sigma_1(t)} \geq 0 \quad \forall p \in G, \tag{3.1}$$

where

$$M_A(f - p^*) = \{t \in A : |f(t) - p^*(t)| = \|f - p^*\|_A\}.$$

Let

$$\mathcal{U} = \{\mathbf{b}(t) = (\overline{\phi_1(t)}, \dots, \overline{\phi_n(t)})\sigma_1(t) : t \in M(f - p^*)\} \cup \left(\bigcup_{t \in B(p^*)} \mathbf{c}(t) \right), \quad (3.2)$$

where

$$\mathbf{c}(t) = (\overline{\phi_1(t)}, \dots, \overline{\phi_n(t)})\sigma_2(t) \quad \forall t \in B(p^*).$$

Then the main theorem of this section can be stated as follows.

Theorem 3.1. *Let $f \in C(Q)$, $p^* \in P_\Omega$. Then the following four statements are equivalent:*

- (i) p^* is a best restricted range approximation to f from P_Ω ;
- (ii) for $\forall p \in P$,

$$\max \left\{ \max_{t \in M(f-p^*)} \operatorname{Re} p(t)\overline{\sigma_1(t)}, \max_{t \in B(p^*)} \max \operatorname{Re} p(t)\overline{\sigma_2(t)} \right\} \geq 0$$

$$\forall p \in P, \quad (3.3)$$

where $p(t)\overline{\sigma_2(t)}$ means $\{p(t)\bar{\sigma} : \sigma \in \sigma_2(t)\}$;

- (iii) the origin of the space \mathbb{C}^n belongs to the convex hull of the set \mathcal{U} ;
- (iv) there exist sets $A_0 = \{t_1, \dots, t_k\} \subseteq M(f - p^*)$, $B_0 = \{t'_1, \dots, t'_m\} \subseteq B(p^*)$, $\sigma_i \in \sigma_2(t'_i)$, $i = 1, \dots, m$ ($m + 1 \leq k + m \leq 2n + 1$) and positive constants $\lambda_1, \dots, \lambda_k, \lambda'_1, \dots, \lambda'_m$ such that the following condition holds:

$$\sum_{l=1}^k \lambda_l p(t_l)\overline{\sigma_1(t_l)} + \sum_{i=1}^m \lambda'_i p(t'_i)\overline{\sigma_i} = 0 \quad \forall p \in P. \quad (3.4)$$

Proof. (i) \Rightarrow (ii): It is sufficient to prove (ii) for $f \in C(Q) \setminus P_\Omega$. Suppose that p^* is a best approximation to f from P_Ω but condition (3.3) does not hold for some $q \in P$. Let $A \subseteq Q$ be an admissible set for f with respect to P_Ω with $|A| \leq 2n + 1$. Then p^* is a best restricted range approximation to f on A from P_Ω and $\|f - p^*\|_A = \|f - p^*\|$. In addition, $M_A(f - p^*) = A \cap M(f - p^*)$. Set

$$B_A(p^*) = \{t \in A : p^*(t) \in \partial\Omega_t\} = A \cap B(p^*).$$

Thus

$$\operatorname{Re} q(t)\overline{\sigma_1(t)} < 0 \quad \forall t \in M_A(f - p^*), \quad (3.5)$$

$$\max \operatorname{Re} q(t)\overline{\sigma_2(t)} < 0 \quad \forall t \in B_A(p^*). \quad (3.6)$$

Write $q_\lambda = p^* - \lambda q$. It follows from (2.2) and (3.6) that

$$\lim_{\lambda \rightarrow 0^+} \frac{F(q_\lambda(t), t) - F(p^*(t), t)}{\lambda} < 0$$

for all $t \in B_A(p^*)$ so that, for each $t \in B_A(p^*)$, there is $\lambda_t > 0$ such that $q_\lambda(t) \in \text{int } \Omega_t$ for all $0 < \lambda \leq \lambda_t$. Taking into account that $p^*(t) \in \text{int } \Omega_t$ for all $t \in A \setminus B_A(p^*)$, we also have that, for each $t \in A \setminus B_A(p^*)$, there is $\lambda_t > 0$ such that $q_\lambda(t) \in \text{int } \Omega_t$ for all $0 < \lambda \leq \lambda_t$. Set $\lambda_0 = \min_{t \in A} \lambda_t$. Then $\lambda_0 > 0$ and $q_\lambda \in P_{A, \Omega}$ for all $0 < \lambda \leq \lambda_0$. Let $G = \{q_\lambda: 0 \leq \lambda \leq \lambda_0\}$. By Proposition 3.1, it follows from (3.5) that p^* is not a best approximation to f from G . This implies that there is $0 < \bar{\lambda} \leq \lambda_0$ such that $\|f - q_{\bar{\lambda}}\|_A < \|f - p^*\|_A = E(f)$ so that

$$E_A(f) \leq \|f - q_{\bar{\lambda}}\|_A < \|f - p^*\|_A = E(f).$$

This contradicts that A is an admissible set for f with respect to P_Ω . Hence (ii) holds.

(ii) \Rightarrow (iii): Suppose that (ii) holds. By the Linear Inequalities Theorem in [3], it suffices to show that \mathcal{U} is compact in \mathbb{C}^n . For the end, let $\{u_k\}$ be any sequence in \mathcal{U} . With no loss of generality, assume that $u_k = (\overline{\phi_1(t_k)}, \dots, \overline{\phi_n(t_k)})\sigma_k$ with $t_k \in B(p^*)$, $\sigma_k \in \sigma_2(t_k)$ and $t_k \rightarrow t_0 \in B(p^*)$. Note that

$$\max \text{Re } \overline{\partial F(z, t)} u \leq F(z + u, t) - F(z, t).$$

It follows that $\partial F(p^*(t), t)$ is uniformly bounded on Q so that $\{\sigma_k\}$ is bounded. Thus, we may assume that $\sigma_k \rightarrow \sigma_0$. From the definition of the subdifferential it follows that

$$F(z, t_k) \geq F(p^*(t_k), t_k) - \text{Re}(z - p^*(t_k))\overline{\sigma_k} \quad \forall z \in \mathbb{C}.$$

Taking the limit as $k \rightarrow \infty$, we have

$$F(z, t_0) \geq F(p^*(t_0), t_0) - \text{Re}(z - p^*(t_0))\overline{\sigma_0} \quad \forall z \in \mathbb{C}.$$

This implies that $\sigma_0 \in \sigma_2(t_0)$ so that $\{u_k\}$ contains a subsequence which converge to an element in \mathcal{U} . This completes the proof of the compactness of the set \mathcal{U} and so (iii) holds.

(iii) \Rightarrow (iv): Suppose that the origin of the space \mathbb{C}^n belongs to the convex hull of the set \mathcal{U} . Then in view of Caratheodory’s theorem in [3] one can find $A_0 = \{t_1, \dots, t_k\} \subseteq M(f - p^*)$, $B_0 = \{t'_1, \dots, t'_m\} \subseteq B(p^*)$, $\mathbf{c}_s(t'_i) \in \mathbf{c}(t'_i)$, $s = 1, \dots, m_i$; $i = 1, \dots, m$ and positive constants $\lambda_1, \dots, \lambda_k, \lambda'_{is}$, $s = 1, \dots, m_i$, $i = 1, \dots, m$ such that

$$\begin{aligned} \sum_{l=1}^k \lambda_l + \sum_{i=1}^m \sum_{s=1}^{m_i} \lambda'_{is} &= 1, \\ \sum_{l=1}^k \lambda_l \mathbf{b}(t_l) + \sum_{i=1}^m \sum_{s=1}^{m_i} \lambda'_{is} \mathbf{c}_s(t'_i) &= 0, \end{aligned} \tag{3.7}$$

$$k + \sum_{i=1}^m m_i \leq 2n + 1.$$

Assume $\mathbf{c}_s(t'_i) = (\overline{\phi_1(t'_i)}, \dots, \overline{\phi_n(t'_i)})\sigma_{is}$ for some $\sigma_{is} \in \sigma_2(t'_i)$, $s = 1, \dots, m_i$, $i = 1, \dots, m$. It follows from (3.7) that

$$\sum_{l=1}^k \lambda_l p(t_l) \overline{\sigma_1(t_l)} + \sum_{i=1}^m \sum_{s=1}^{m_i} \lambda'_{is} p(t'_i) \overline{\sigma_{is}} = 0 \quad \forall p \in P. \tag{3.8}$$

Set

$$\lambda'_i = \sum_{s=1}^{m_i} \lambda'_{is}, \quad \sigma_i = \frac{\sum_{s=1}^{m_i} \lambda'_{is} \sigma_{is}}{\lambda'_i}, \quad i = 1, \dots, m.$$

Then, due to the convexity of $\sigma_2(t'_i)$, $\sigma_i \in \sigma_2(t'_i)$. From (3.8) we have (3.4). The fact that $k \geq 1$ follows from

$$\min \operatorname{Re}(p_0(t'_i) - p^*(t'_i)) \overline{\sigma_2(t'_i)} > 0, \quad i = 1, \dots, m,$$

by Proposition 2.1, where p_0 satisfies (2.1). The proof of implication (iii) \Rightarrow (iv) is complete.

(iv) \Rightarrow (i): Suppose that $A_0 = \{t_1, \dots, t_k\} \subseteq M(f - p^*)$, $B_0 = \{t'_1, \dots, t'_m\} \subseteq B(p^*)$, $\sigma_i \in \sigma_2(t'_i)$, $i = 1, \dots, m$ ($m + 1 \leq k + m \leq 2n + 1$) and positive constants $\lambda_1, \dots, \lambda_k, \lambda'_1, \dots, \lambda'_m$ such that (3.4) holds. With no loss of generality, assume that $\sum_{l=1}^k \lambda_l = 1$. For any $p \in P_\Omega$, one has by Proposition 2.1 that

$$\operatorname{Re}(p^* - p)(t'_i) \overline{\sigma_i} \leq 0, \quad i = 1, 2, \dots, m,$$

and it follows that

$$\begin{aligned} \|f - p\|^2 &\geq \sum_{l=1}^k \lambda_l |(f - p)(t_l)|^2 + 2\operatorname{Re} \sum_{i=1}^m \lambda'_i (p^* - p)(t'_i) \overline{\sigma_i} \\ &= \sum_{l=1}^k \lambda_l |(f - p^*)(t_l)|^2 + \sum_{i=1}^m \lambda_i |(p^* - p)(t'_i)|^2 \\ &\geq \|f - p^*\|^2, \end{aligned}$$

where the equality holds because of (3.4). This means that p^* is a best approximation to f from P_Ω and hence (i) holds. The proof of Theorem 3.1 is complete. \square

4. Uniqueness and strong uniqueness of the best approximation

In order to establish some results on the uniqueness and strong uniqueness of the best approximation from P_Ω , we introduce the concept of n -dimensional Haar spaces of $C(Q)$ taken from [4].

Definition 4.1. An n -dimensional subspace $P \subset C(Q)$ is called a Haar space if every element $p \in P \setminus \{0\}$ has at most $n - 1$ zeros in Q .

In the rest of this section we always assume that P is an n -dimensional Haar space.

As illustrated by the example given in [10], the best approximation to f from P_Q may not be unique, in general, even in the case when P is an n -dimensional Haar space. Hence, in [10], the admissible family of $C(Q)$ was introduced to discuss the uniqueness problem.

Definition 4.2 (Smirnov and Smirnov [10]). A function $f \in C(Q)$ is called admissible if

$$f(t) \in \Omega_t \quad \forall t \in Q \tag{4.1}$$

or there exists a best approximation p^* to f from P_Q such that

$$M(f - p^*) \cap B(p^*) = \emptyset. \tag{4.2}$$

The set of all admissible functions is denoted by $C_a(Q)$.

Lemma 4.1. *Suppose that $f \in C_a(Q) \setminus P_Q$ and $p^* \in P_Q$ is a best approximation to f from P_Q . Let $A_0 = \{t_1, \dots, t_k\} \subseteq M(f - p^*)$, $B_0 = \{t'_1, \dots, t'_m\} \subseteq B(p^*)$ satisfy (3.4). If at least one of conditions (4.1) and (4.2) holds, then $|A_0 \cup B_0| \geq n + 1$.*

Proof. Without loss of generality, assume

$$A_0 \setminus B_0 = \{t_{r+1}, \dots, t_k\}, \quad B_0 \setminus A_0 = \{t'_{r+1}, \dots, t'_m\},$$

$$A_0 \cap B_0 = \{t_1, \dots, t_r\} = \{t'_1, \dots, t'_r\}.$$

Then, by Proposition 2.1, $\min \operatorname{Re} \overline{\sigma_1(t)} \sigma_2(t) \geq 0, \quad \forall t \in A_0 \cap B_0$ if condition (4.1) holds. In addition, it is trivial that $A_0 \cap B_0 = \emptyset$ if condition (4.2) holds. Suppose on the contrary that $|A_0 \cup B_0| \leq n$. Then there exists $q \in P$ such that

$$q(t_l) = \sigma_1(t_l) \quad \forall l = r + 1, \dots, k,$$

$$q(t'_i) = \sigma_i \quad \forall i = r + 1, \dots, m,$$

$$q(t_l) = q(t'_l) = \sigma_1(t_l) + \sigma_l \quad \forall l = 1, \dots, r,$$

where $\sigma_i \in \sigma_2(t'_i), \quad i = 1, \dots, m$ satisfy (3.4). Obviously,

$$\begin{aligned} & \sum_{l=1}^k \lambda_l q(t_l) \overline{\sigma_1(t_l)} + \sum_{i=1}^m \lambda'_i q(t'_i) \overline{\sigma_i} \\ &= \sum_{l=1}^r \lambda_l (|\sigma_1(t_l)|^2 + \sigma_l \overline{\sigma_1(t_l)}) + \sum_{l=r+1}^k \lambda_l |\sigma_1(t_l)|^2 \\ & \quad + \sum_{i=1}^r \lambda'_i (|\sigma_i|^2 + \sigma_1(t_i) \overline{\sigma_i}) + \sum_{i=r+1}^m \lambda'_i |\sigma_i|^2 \\ & > 0, \end{aligned}$$

which contradicts (3.4) and completes the proof. \square

Lemma 4.2. *Suppose that $f \in C_a(Q) \setminus P_\Omega$ and $p^* \in P_\Omega$ is any best approximation to f from P_Ω . Then $|M(f - p^*) \cup B(p^*)| \geq n + 1$.*

Proof. From Lemma 4.1, it suffices to show the conclusion of Lemma 4.2 remains true when condition (4.1) does not hold. In this case, there exists one best approximation p_0^* to f from P_Ω such that $M(f - p_0^*) \cap B(p_0^*) = \emptyset$. Let $\bar{p} = (p^* + p_0^*)/2$. Then $\bar{p} \in P_\Omega$ is also a best approximation to f from P_Ω . Using standard techniques, we get the inclusions

$$M(f - \bar{p}) \subseteq M(f - p_0^*) \cap M(f - p^*),$$

$$B(\bar{p}) \subseteq B(p_0^*) \cap B(p^*).$$

This implies that

$$M(f - \bar{p}) \cap B(\bar{p}) \subseteq M(f - p_0^*) \cap B(p_0^*) = \emptyset,$$

so that $|M(f - \bar{p}) \cup B(\bar{p})| \geq n + 1$ due to Lemma 4.1. Consequently, $|M(f - p^*) \cup B(p^*)| \geq n + 1$. The proof is complete. \square

Recall that a convex subset J of \mathbb{C} is strictly convex if, for any two distinct elements $z_1, z_2 \in J$, $\frac{1}{2}(z_1 + z_2) \in \text{int } J$.

Theorem 4.1. *Suppose that Ω_t is strictly convex for each $t \in Q$. Then each $f \in C_a(Q)$ has a unique best approximation to f from P_Ω .*

Proof. The case when $f \in P_\Omega$ is trivial. Now let $f \in C_a(Q) \setminus P_\Omega$. Suppose on the contrary that f has two distinct best approximation p_1, p_2 from P_Ω . Let $p^* = (p_1 + p_2)/2$. Then p^* is also a best approximation to f from P_Ω . Set $Z(p) = \{t \in Q : p(t) = 0\}$. We have that

$$M(f - p^*) \subseteq M(f - p_1) \cap M(f - p_2) \subseteq Z(p_1 - p_2)$$

and

$$B(p^*) \subseteq B(p_1) \cap B(p_2).$$

This implies that $B(p^*) \subseteq Z(p_1 - p_2)$ by the strict convexity of Ω_t . Lemma 4.2 implies that $p_1 - p_2$ has at least $n + 1$ zeros so that $p_1 = p_2$ in view of the definition of a Haar space. This completes the proof. \square

Remark 4.1. The strict convexity of Ω_t in Theorem 4.1 cannot be dropped as shown in the following example.

Example 4.1. Let $Q = \{-1, 0, 1\}$, $p_1(t) = 1$, $p_2(t) = t$. Let $\Omega_1 = \Omega_{-1} = \{z : \text{Re } z \geq 1\}$ and $\Omega_0 = \mathbb{C}$. Then $P = \text{span}\{p_1, p_2\}$ is a Haar subspace. Clearly,

$$P_\Omega = \{p = \alpha + \beta t \in P : \text{Re } (\alpha + \beta) \geq 1, \text{Re } (\alpha - \beta) \geq 1\}.$$

Now define $f \in C(Q)$ by

$$f(-1) = f(1) = \frac{3}{2}, \quad f(0) = 0$$

and take $p^* \equiv 1$. Then $\|f - p^*\| = 1$ and

$$M(f - p^*) = \{0\}, \quad B(p^*) = \{-1, 1\}.$$

We will show that p^* is a best approximation to f from P_Ω . In fact, $\forall p = \alpha + \beta t \in P_\Omega$, by the definition,

$$\operatorname{Re}(\alpha + \beta) \geq 1, \quad \operatorname{Re}(\alpha - \beta) \geq 1.$$

It follows that

$$\operatorname{Re}(\alpha + \beta) + \operatorname{Re}(\alpha - \beta) \geq 2,$$

which implies that $\operatorname{Re} \alpha \geq 1$. Thus,

$$\|f - p\| \geq |(f - p)(0)| = |\alpha| \geq \operatorname{Re} \alpha \geq 1.$$

This shows that p^* is a best approximation to f from P_Ω . Hence f is admissible since $M(f - p^*) \cap B(p^*) = \emptyset$

On the other hand, if $\bar{p} = 1 + \frac{i}{2}t$, it is easy to verify that $\bar{p} \in P_\Omega$ and

$$\|f - \bar{p}\| = |(f - \bar{p})(0)| = 1.$$

This implies that \bar{p} is also a best approximation to f from P_Ω . \square

Now let us consider the strong uniqueness of the best approximation to f from P_Ω . We first give the definition of the strong uniqueness of order $\alpha > 0$, see, for example, [7,8].

Definition 4.3. Suppose that $f \in C(Q)$ and $p^* \in P_\Omega$ is a best approximation to f from P_Ω . p^* is called strongly unique of order $\alpha > 0$ if there exists a constant $c_\alpha = c_\alpha(f) > 0$ such that

$$\|f - p\|^\alpha \geq \|f - p^*\|^\alpha + c_\alpha \|p - p^*\|^\alpha \quad \forall p \in P_\Omega.$$

The following lemma extends Theorem 3.3 of [11,12].

Lemma 4.3. Suppose that $f \in C_a(Q) \setminus P_\Omega$ and $p^* \in P_\Omega$ is a unique best approximation to f from P_Ω . Let $r > 0$. If, for each $t \in B(p^*)$, there exist a neighborhood $U_t(p^*(t))$ of $p^*(t)$ and a positive constant γ_t such that

$$\max \operatorname{Re}(p^*(t) - z) \bar{\sigma}_2(t) \leq -\gamma_t |z - p^*(t)|^r \quad \forall z \in \Omega_t \cap U_t(p^*(t)), \tag{4.3}$$

then p^* is strongly unique of order $\alpha = \max\{2, r\}$.

Proof. Since $p^* \in P_\Omega$ is a best approximation to f from P_Ω , it follows from Theorem 3.1(iv) that there exist sets $A_0 = \{t_1, \dots, t_k\} \subseteq M(f - p^*)$, $B_0 = \{t'_1, \dots, t'_m\} \subseteq B(p^*)$, $\sigma_i \in \sigma_2(t'_i)$, $i = 1, \dots, m$ ($m + 1 \leq k + m \leq 2n + 1$) and positive constants $\lambda_1, \dots, \lambda_k, \lambda'_1, \dots, \lambda'_m$ with $\sum_{l=1}^k \lambda_l = 1$ such that (3.4) holds. From Lemma 4.1 we have that $|A_0 \cup B_0| \geq n + 1$.

For any $p \in P$, define

$$\|p\|_\alpha = \left(\sum_{l=1}^k \lambda_l |p(t_l)|^\alpha + \sum_{i=1}^m \lambda'_i |p(t'_i)|^\alpha \right)^{1/\alpha}.$$

Then $\|\cdot\|_\alpha$ is a norm equivalent to the uniform norm so that there exists a constant $\eta > 0$ such that

$$\|p\|_\alpha \geq \eta \|p\| \quad \forall p \in P.$$

Set

$$\gamma_\alpha(p) = \frac{\|f - p\|^\alpha - \|f - p^*\|^\alpha}{\|p - p^*\|^\alpha} \quad \forall p \in P_\Omega, \quad p \neq p^*.$$

Then $\gamma_\alpha(p)$ has positive lower bounds on $P_\Omega \setminus \{p^*\}$. In fact, if otherwise, there exists a sequence $\{p_j\} \subset P$ such that $\gamma_\alpha(p_j) \rightarrow 0$. Then $\|f - p_j\| \rightarrow \|f - p^*\|$. With no loss of generality, we may assume that $p_j \rightarrow p^*$ due to the uniqueness of the best approximation. Write $d_r = \min_{1 \leq i \leq m} \gamma_{t_i} > 0$. From (3.4) and (4.3), we have that

$$\begin{aligned} \|f - p_j\|^2 &\geq \sum_{l=1}^k \lambda_l |f(t_l) - p_j(t_l)|^2 + 2 \sum_{i=1}^m \lambda'_i \operatorname{Re}(p^*(t'_i) - p_j(t'_i)) \bar{\sigma}_i \\ &\quad + 2d_r \sum_{i=1}^m \lambda'_i |p^*(t'_i) - p_j(t'_i)|^r \\ &= \|f - p^*\|^2 + \sum_{l=1}^k \lambda_l |p_j(t_l) - p^*(t_l)|^2 + 2d_r \sum_{i=1}^m \lambda'_i |p_j(t'_i) - p^*(t'_i)|^r \\ &\geq \|f - p^*\|^2 + \sum_{l=1}^k \lambda_l |p_j(t_l) - p^*(t_l)|^\alpha + 2d_r \sum_{i=1}^m \lambda'_i |p_j(t'_i) - p^*(t'_i)|^\alpha \\ &\geq \|f - p^*\|^2 + \min\{1, 2d_r\} \|p_j - p^*\|^\alpha \\ &\geq \|f - p^*\|^2 + \min\{1, 2d_r\} \eta^\alpha \|p_j - p^*\|^\alpha \end{aligned}$$

for all j large enough. Observe that

$$\|f - p_j\|^\alpha - \|f - p^*\|^\alpha \geq (\alpha/2) \|f - p^*\|^{\alpha-2} (\|f - p_j\|^2 - \|f - p^*\|^2).$$

It follows that $\gamma_\alpha(p_j) \geq \min\{1, 2d_r\} (\alpha/2) \|f - p^*\|^{\alpha-2} \eta^\alpha > 0$, which contradicts that $\gamma_\alpha(p_j) \rightarrow 0$. The proof is complete. \square

The following result is a generalization of Theorem 5.2 in [10].

Theorem 4.2. *Suppose that $\partial\Omega_t$ has a positive curvature at z^* for any $t \in Q$, $z^* \in \partial\Omega_t$. Then each $f \in C_a(Q)$ has a strongly unique best approximation of order 2 from P_Ω .*

Proof. The case when $f \in P_\Omega$ is trivial so that we assume that $f \notin P_\Omega$. Note that each Ω_t is strictly convex under the assumption of Theorem 4.2. By Theorem 4.1, the best approximation p^* to f from P_Ω is unique. By Lemma 4.3, it is sufficient to show that, for each $t \in B(p^*)$, there exist a neighborhood $U_t(p^*(t))$ of $p^*(t)$ and a positive constant $\gamma_t > 0$ such that (4.3) holds for $r = 2$.

For each $t \in B(p^*)$, let $\kappa_t > 0$ and $u(t)$ denote the curvature and center of curvature at $p^*(t)$, respectively. Define

$$c(t) = 2u(t) - p^*(t), \quad r(t) = 2|u(t) - p^*(t)| = 2/\kappa_t \quad \forall t \in B(p^*).$$

Then, for each $t \in B(p^*)$, there exists a neighborhood $U_t(p^*(t))$ of $p^*(t)$ such that

$$|z - c(t)| \leq r(t) \quad \text{for all } z \in \Omega_t \cap U_t(p^*(t)). \tag{4.4}$$

From (4.4), we obtain that

$$\begin{aligned} \operatorname{Re}(p^*(t) - z)\overline{(c(t) - p^*(t))} &\leq -\frac{1}{2}|z - p^*(t)|^2, \\ z &\in \Omega_t \cap U_t(p^*(t)), \quad t \in B(p^*). \end{aligned} \tag{4.5}$$

Observe that, for any $t \in B(p^*)$ and $\sigma \in \sigma_2(t)$, $\sigma = d_t(c(t) - p^*(t))$ for some $d_t > 0$. This with (4.5) implies that

$$\begin{aligned} \max \operatorname{Re}(p^*(t) - z)\overline{\sigma_2(t)} &= d_t \operatorname{Re}(p^*(t) - z)\overline{c(t) - p^*(t)} \\ &\leq -\frac{d_t}{2}|z - p^*(t)|^2 \end{aligned}$$

for any $z \in \Omega_t \cap U_t(p^*(t))$; $t \in B(p^*)$. This completes the proof. \square

In order to give the more general strong uniqueness theorems, we introduce the notation of uniformly convex function and some useful properties, see, for example, [16].

Definition 4.4. A function $F : \mathbb{C} \rightarrow \mathbb{R}$ is uniformly convex at $z^* \in \mathbb{C}$ if there exists $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\delta(x) > 0$ for $x > 0$ such that

$$\begin{aligned} F(\lambda z^* + (1 - \lambda)z) &\leq \lambda F(z^*) + (1 - \lambda)F(z) - \lambda(1 - \lambda)\delta(|z^* - z|) \\ \forall z \in \mathbb{C}, \quad 0 &< \lambda < 1. \end{aligned}$$

Note that the set $\{z \in \mathbb{C} : F(z) \leq 0\}$ is strictly convex if F is uniformly convex at each $z^* \in \mathbb{C}$ with $F(z^*) = 0$. Define the modulus of convexity of F at z^* as follows:

$$\begin{aligned} \mu_{z^*}(x) &= \inf \left\{ \frac{\lambda F(z^*) + (1 - \lambda)F(z) - F(\lambda z^* + (1 - \lambda)z)}{\lambda(1 - \lambda)} : \right. \\ &\quad \left. z \in \mathbb{C}, \quad |z^* - z| = x, \quad 0 < \lambda < 1 \right\}. \end{aligned}$$

Clearly, F is uniformly convex at z^* if and only if $\mu_{z^*}(x) > 0$ for all $x > 0$.

Definition 4.5. A function $F : \mathbb{C} \rightarrow \mathbb{R}$ has the modulus of convexity of order $r > 0$ at $z^* \in \mathbb{C}$ if there exists $d_r > 0$ such that $\mu_{z^*}(x) > d_r x^r$ for $x > 0$.

Proposition 4.1. A function $F : \mathbb{C} \rightarrow \mathbb{R}$ has the modulus of convexity of order $r > 0$ at $z^* \in \mathbb{C}$ if and only if there exists $d > 0$ such that

$$F(z) \geq F(z^*) + \operatorname{Re}(z - z^*)\bar{u} + d|z - z^*|^r \quad \forall z \in \mathbb{C}, u \in \partial F(z^*). \quad (4.6)$$

Theorem 4.3. Let $r > 0$. Suppose that, for any $t \in Q$, $z^* \in \partial\Omega_t$, $F(\cdot, t)$ has the modulus of convexity of order r at z^* . Then each $f \in C_a(Q)$ has a strongly unique best approximation of order $\alpha = \max\{r, 2\}$ to f from P_Ω .

Proof. Since for any $t \in Q$, $z^* \in \partial\Omega_t$, $F(\cdot, t)$ is uniformly convex at z^* , it follows that each Ω_t is strictly convex. Thus, by Theorem 4.1, the best approximation p^* to f from P_Ω is unique. By the assumption, for each $t \in B(p^*)$, there exists $\eta_t > 0$ such that (4.6) holds for $d = d_t$. This implies that (4.3) holds. Thus the result follows from Lemma 4.3. The proof is complete. \square

Remark 4.2. In the case when F has the continuous second derivatives, we can show that the fact that, for each $t \in Q$, $\partial\Omega_t$ has a positive curvature at each $z^* \in \partial\Omega_t$ implies that $F(\cdot, t)$ has the modulus of convexity of order 2 at z^* for any $t \in Q$, $z^* \in \partial\Omega_t$. Hence, in this case, Theorem 4.2 is a direct corollary of Theorem 4.3.

Remark 4.3. When Ω_t is the closed disk in \mathbb{C} , the assumptions of Theorems 4.2 and 4.3 hold. Hence they extend the strong uniqueness theorem in [10].

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